Group Report 1964-63

R. T. Prosser

An Inequality for Certain Correlation Functions

29 October 1964

Prepared under Electronic Systems Division Contract AF 19(628)-500 by

Lincoln Laboratory

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Lexington, Massachusetts

The work reported in this document was performed at Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology, with the support of the U.S. Air Force under Contract AF 19(628)-500.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
LINCOLN LABORATORY

AN INEQUALITY FOR CERTAIN CORRELATION FUNCTIONS

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Group 66

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ABSTRACT

An inequality for certain types of generalized auto correlation functions, of interest in the study of varactor diodes, is herein established.

Accepted for the Air Force
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An Inequality for Certain Correlation Functions

A study of the properties of varactor diodes has recently led to the following problem whose solution is set out below: Let $q(t)$ be a continuous function, periodic with period 2π. It is known that the auto-correlation function $R(t)$ associated with $q(t)$, as defined by

$$R(t) = \int_{0}^{2\pi} q(t - t')\, q(t')\, dt' \tag{1}$$

has the property that it achieves its maximum at the origin:

$$R(t) \leq R(0) \quad . \tag{2}$$

Now suppose $U(x)$ and $V(x)$ are continuous monotone increasing functions, defined at least on the range of $q(t)$. Consider the modified correlation function $S(t)$, defined by

$$S(t) = \int_{0}^{2\pi} U(q(t - t'))\, V(q(t'))\, dt' \tag{3}$$

Question: does $S(t)$ achieve its maximum at the origin, i.e., does (2) hold for $S(t)$?

We shall show here that the answer is affirmative. The proof depends

on a simple inequality found in Hardy and Littlewood [1, page 261].

Lemma   Let $\{a_i\}$ and $\{b_i\}$ be two finite sequences of real numbers, both arranged in decreasing order. Then for any permutations π and σ of the integers, we have

$$\sum_{i=1}^{n} a_{\pi(i)} b_{\sigma(j)} \leq \sum_{i=1}^{n} a_i b_i \tag{4}$$

Proof   It suffices to consider the case where $a_{\pi(i)} = a_i$. Then either $b_{\sigma(j)} = b_j$, or else for some j and k we have $j < k$ and $b_{\sigma(j)} < b_{\sigma(k)}$. Then we have

$$(a_j b_{\sigma(k)} + a_k b_{\sigma(j)}) - (a_j b_{\sigma(j)} + a_k b_{\sigma(k)})$$

$$= (a_j - a_k)(b_{\sigma(k)} - b_{\sigma(j)}) \geq 0 \tag{5}$$

Hence we will not diminish the sum $\sum a_i b_{\sigma(i)}$, by exchanging $b_{\sigma(j)}$ and $b_{\sigma(k)}$. A finite number of such exchanges leads to a new permutation σ' with $b_{\sigma'(i)} = b_i$, and a sum no smaller than the original.

To show that (2) holds for $S(t)$ defined by (3), we simply approximate the continuous function $q(t)$ uniformly by a step function $r(t)$ defined so that

$$r(t) = q\left(\frac{2i\pi}{n}\right) = q_i \quad \text{when} \quad \frac{(2i-1)\pi}{n} \leq t < \frac{(2i+1)\pi}{n} \quad i = 0, 1, \ldots, n\text{-}1 \tag{6}$$

By choosing n sufficiently large, we can arrange so that

$$| q(t) - r(t) | < \varepsilon \qquad \text{for all } t \qquad 0 \leq t < 2\pi \tag{7}$$

Since U and V are continuous, we can also arrange so that

$$| U(q(t)) - U(r(t)) | < \varepsilon$$

$$| V(q(t)) - V(r(t)) | < \varepsilon \qquad \text{for all } t,\ 0 \leq t < 2\pi \tag{8}$$

Using $r(t)$, we define the function $T(t)$ by

$$T(t) = \int_{o}^{2\pi} U(r(t - t')) \; V(r(t')) \; dt$$

$$= \sum_{j=1}^{n} U(q_{i-j}) \; V(q_j) \qquad \text{if } t = \frac{2i\pi}{n} \tag{9}$$

Now the inequality of the Lemma tells us that

$$T(t) \leq T(0) \qquad \text{for } t = \frac{2i\pi}{n} \tag{10}$$

On the other hand, we have

$$\left| S(t) - T(t) \right| \le \int_0^{2\pi} \left| U(q(t-t')) - U(r(t-t')) \right| \left| V(q(t')) \right| dt'$$

$$+ \int_0^{2\pi} \left| U(r(t-t')) \right| \left| V(q(t')) - V(r(t')) \right| dt'$$

$$\le 4\pi A\varepsilon \qquad (11)$$

where A is bigger than the maximum value attained by $\left| U(q(t)) \right|$ or $\left| V(q(t)) \right|$ as t ranges from 0 to 2π. Combining (10) and (11), we obtain

$$S(t) \le S(0) + 8\pi A\varepsilon \qquad \text{for } t = \frac{2i\pi}{n} \qquad (12)$$

Since this inequality must hold for all choices of n, we conclude that

$$S(t) \le S(0) \qquad \text{for all } t, \qquad 0 \le t < 2\pi \qquad (13)$$

BIBLIOGRAPHY

1. Hardy Littlewood and Polya, Inequalities, Cambridge University Press (1959).

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DOCUMENT CONTROL DATA - R&D

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1. ORIGINATING ACTIVITY *(Corporate author)* Lincoln Labs., Lexington, Mass.	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP N/A

3. REPORT TITLE

An Inequality for Certain Correlation Functions

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Group Report

5. AUTHOR(S) *(Last name, first name, initial)*

Prosser, R.T.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
Oct 64	8	0

8a. CONTRACT OR GRANT NO. AF19(628)500	9a. ORIGINATOR'S REPORT NUMBER(S) GR-1964-63
b. PROJECT NO.	
c.	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*
d.	ESD-TDR-64-566

10. AVAILABILITY/LIMITATION NOTICES

Qualified Requesters May Obtain from DDC.
Aval from OTS.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	ESD, L.G. Hanscom Field, Bedford, Mass.

13. ABSTRACT

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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Diodes Correlation Techniques Statistical Analysis Inequalities						

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